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Length of a Chord and Side of a Regular Polygon in a Circle

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ABSTRACT

This paper aims to present a dimensional analysis-based approach for the derivation of Bhāskara II's formula for the length of a chord in a circle. A restated and simplified formula of Bhāskara II for the chord length is also presented. Further, a formula for the length of the side of a regular polygon in a circle is derived. Two similar problems from *Lilāvatī* and *Gaṇita Kaumudī* are worked out and their special features are brought out.

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1.0 INTRODUCTION

The circle occupied an important place in the mathematical formulations even during the *Śulbasūtra* days. The value of Π was known approximately. Jain Geometry devoted considerable attention to the calculation of the circumference, area, arc length, segment height, chord length, etc.

The *Āryabhaṭīyam* of Āryabhaṭa (5th Cent. C.E.) gives a very good approximation for Π and a table for sines with intervals of 225 minutes of arc for the R-sine differences.

In *Mahābhāskarīyam*, Bhāskara I (7th Cent. C.E.) has given a very close approximation for the R sine of an acute angle, without having to use the sine table. His formula is:

$$R\ sin\theta = \frac{4R(180-\theta)\theta}{40500-\theta(180-\theta)}$$

Where θ is in degrees and R is the radius. If θ is in radians, the formula takes the form:

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$$R\ sin\theta = \frac{160(\pi - \theta)}{5\pi^2 - 4\theta(\pi - \theta)}$$ [1]

It is this formula that has paved the way for Bhāskara II's formula for the length of the chord of a circle.

1.1 Bhāskara II's Formula for the Length of a Chord

For a circle with diameter *d*, arc length '*a*' and circumference '*p*', Bhāskara says:

चापोननिघ्नपरिधिः प्रथमाहव्यः स्यात्
पंचाहतः परिधिवर्गचतुर्थभागः ।
आद्योनितेन खलु तेन भजेच्चतुर्घ्न-
व्यासहतं प्रथमप्राप्तमिह ज्यका स्यात् ॥

(*Lī.* CCXIX)

The circumference diminished and multiplied by the arc shall be called the first (*Prathamā*). One quarter of the square of the circumference multiplied by 5 is to be diminished by the *Prathamā*. The *Prathamā* multiplied by 4 times the diameter should be divided by the above difference. The quotient will be the chord.

In mathematical terms, the rule becomes:

$$c = \frac{4d.a(p - a)}{\frac{5}{4}p^2 - a(p - a)}$$

Where c = Chord length, d = Diameter, a = Arc length, and p = Circumference

1.2 Proof for the Chord Formula

R. C. Gupta outlines five different approaches (Two by earlier authors and three by Gupta himself) for the derivation of the formula in the π - θ format.

Gaṇeśa Daivajña, the commentator of '*Lilāvatī*' in his *Buddhivilāśanī*, gives a proof of the formula, but at the same time points out that the denominator of the formula has to be somehow guessed, for using the Rule of Three, for the derivation.

"पञ्चाहतेन परिघिवर्गचतुर्थांशेन
इष्टप्रथमोनेनेष्टहर ऊहनीयः ।
अत उक्तं – पञ्चाहतः परिघिवर्गचतुर्थेत्यादि ।
इदं जीवा साधनार्थं यथा कथञ्चित्
त्रैराशिकमुपलब्ध्याऽऽचार्यैः कल्पितम् ॥"

1.3 Proof Based on Dimensional Analysis Approach

1.3.1 Dimensional Analysis

A dimensional formula is the expression of a physical quantity in terms of the fundamental dimensions. While there are seven fundamental or base quantities *viz.,* mass, length, time,

[1] The relevant Sanskrit verse along with the translation is given in reference 4 by R. C. Gupta.

temperature, electric, current, luminous intensity and amount of substance the ones most regularly used are mass, length, and time, represented by *M*, *L,* and *T*.

If we want to represent velocity for example, it is distance/time $= L/T = LT^{-1}$ in dimensional terms. Similarly, acceleration is velocity/time, *i.e.,* $LT^{-1}/T = LT^{-2}$.

The dimensional analysis has been used mainly in the area of Fluid mechanics, but it is equally applicable to other sciences also. This approach helps us to:

1. Check the consistency of formulas relating to physical quantities by checking the equality of dimensions on both sides of a formula.

   Taking a simple example, the equation $F = ma$ represents the Force of a body of mass m, moving with an acceleration of a.

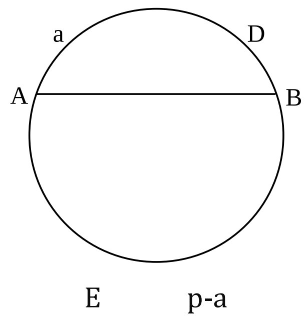
   The unit of force is meter kg/sec^2 which is dimensionally $LM/T^2 = LMT^{-2}$. The dimensions of ma are also $ML/T^2 = LMT^{-2}$. The equation $F = ma$ is thus consistent.

2. Derive, where possible, the formula for a physical quantity once its dependence on other quantities is known.

1.4 Derivation of Formula for the Length of a Chord

The length of a chord depends on (1) the diameter of the circle. The chord length increases or decreases according to the diameter. Thus $c \alpha d$.

(i) The factor a (p – a)



In the circle shown, the chord AB can be considered to correspond to either the minor arc *a (A D B)* or the major arc *(p-a) (A E B)*.

When arc length $a = 0$, chord length $c = 0$ when $a = p/2$, chord length becomes d. Thereafter, as the arc length increases beyond $p/2$, chord length decreases and becomes 0 when *arc length = p*.

To satisfy the above constraints, chord length should depend on $a\,(p - a)$.

Now, the arc can be considered as a fraction of the circumference. Let $a = p/n$, where n is a positive integer.

Therefore $a\,(p - a)$ can be written as

$$\frac{p}{n}\left(p - \frac{p}{n}\right) = \frac{p^2}{n^2}(n-1)$$

Thus, we get

$$c \propto d \times \frac{p^2}{n^2}(n-1)$$

However, as the dimension of c viz., L is already balanced by d, $p^2/n^2\ (n - 1)$ introduces an extra dimension of L^2. For balancing the dimensions, we must introduce a denominator, with dimension of L^2.

The denominator takes the form of $K_1P^2 + K_2P^2\ (n-1)/n^2$ where K_1 and K_2 are constants, either of which can be positive or negative. The extra factor K_1P^2 helps in avoiding the situation where denominator is zero when $n = 1$, maintains the dimensions of the denominator as L^2, and renders the numerator/denominator, a dimensionless constant as required. Thus, chord:

$$c = \frac{d \times \left(\frac{n-1}{n^2}\right) p^2}{k_1 p^2 + \left(k_2 \frac{n-1}{n^2}\right) p^2}$$

$$c = \frac{d(n-1)}{k_1 n^2 + k_2(n-1)} \qquad \text{------------ } \mathbf{(1)}$$

To evaluates the constants K_1 & K_2, we proceed as follows:

When $a = \frac{p}{2}\ c = d.n = 2$ here

Thus equation (1) becomes

$$d = \frac{d(2-1)}{k_1(4) + k_2 \cdot 1}$$

$$d = \frac{d}{4k_1 + k_2}$$

Or

$$4k_1 + k_2 = 1 \qquad \text{------------ } \mathbf{(2)}$$

(ii) When a = p/6 and c = d/2

This result was known from early days, for, Āryabhaṭa says (Ab., gaṇitapāda, 9):

"परिधेः षड्भागज्या विष्कम्भार्धेन सा तुल्या ।"

"The chord of one–sixth of the circumference is equal to half the diameter"

Therefore

$$c = \frac{d}{2} = \frac{d.(6-1)}{k_1(36) + k_2(5)}$$

$$\frac{1}{2} = \frac{5}{36k_1 + 5k_2}$$

$$36k_1 + 5k_2 = 10 \qquad \text{------------ } \mathbf{(3)}$$

Multiplying (2) by 9 yields

$$36k_1 + 4k_2 = 9 \qquad \text{------------ } \mathbf{(4)}$$

$$(3) - (4) \text{ gives}$$

$$-4k_2 = 1$$

Therefore

$$k_2 = -\frac{1}{4}$$

Substituting in (2)

$$4k_1 - \frac{1}{4} = 1$$

$$4k_1 = \frac{5}{4}$$

$$k_1 = \frac{5}{16}$$

Thus, we have

$$C = \frac{d.(n-1)}{\frac{5}{16}n^2 - \frac{1}{4}(n-1)}$$

$$C = \frac{4d(n-1)}{\frac{5}{4}n^2 - (n-1)} \qquad \text{------------ } \mathbf{(5)}$$

Multiplying both numerator and denominator by p^2/n^2, we get

$$c = \frac{4d\frac{n-1}{n^2}p^2}{\frac{5}{4}p^2 - \frac{n-1}{n^2}p^2}$$

$$c = \frac{4da(p-a)}{\frac{5}{4}p^2 - a(p-a)} \qquad \text{------------ } \mathbf{(6)}$$

This is Bhāskara II's formula.

We can use equations (5) and (6) in solution of problems. However (5) is more advantageous and easier to work with as will be shown below.

1.5 Advantages in Using the Chord-Length Equation (6)

Bhāskara II's formula for chord length is:

$$c = \frac{4da(p-a)}{\frac{5}{4}p^2 - a(p-a)}$$

In solving problems, the above formula if used in its present form, has a drawback in the form of rounding-off errors as indicated below:

1. In determining p using the relation *Π = 3927/1250* and rounding off.
2. The factor *a (p – a)* is again rounded off every time "a" varies.
3. Similarly, there is rounding-off error in calculating the denominator.

These difficulties are avoided in using the following formula:

$$c = \frac{4d(n-1)}{\frac{5}{4}n^2 - (n-1)}$$

Where *n* is simply the number of arcs each of length *a* contained in *p* and no stage-wise rounding off is involved.

1.6 Side of a Regular Polygon Drawn within a Circle

Consider a regular polygon of *n* sides drawn within *a* circle of diameter *d*. This implies there are *n* arcs of equal length corresponding to the *n* sides of the polygon, each side of the polygon being a chord of the arc. We can therefore use formula (5) viz.

$$c = \frac{4d(n-1)}{\frac{5}{4}n^2 - (n-1)}$$

Given d and n, where 'n' indicates the number of sides of a regular polygon inside the circle.

1.7 Examples from *Gaṇita Kaumudī* and *Lilāvatī*:

We consider below, two interesting examples form *Gaṇita Kaumudī* and *Lilāvatī*. In both the examples we are asked to find the chord lengths for arcs of different lengths for a given circle.

पञ्चाशता सङ्गुणितानि यत्र नवैकपूर्वाणि धनूंषि विद्वन् ।
व्यासः खखाग्निप्रमितास्त्रिनिघ्ना वृत्तिः पृथक् पृथक् तत्र वदांशु जीव ॥

(G.K. Udāharaṇa 58)

Where the arcs are obtained by multiplying 50 successively by numbers from 1 to 9, The diameter is 300. The circumference is three times this value *i.e.,* 900, O learned, tell me the chord lengths.

Solution:

Here the number of arcs = 9, a = 50, 100, 150, 200, 250, 300, 350, 400 and 450. Given p = 900, the arcs are simply: $p/18$, $2p/18$, $3p/18$, $4p/18$, $5p/18$, $6p/18$, $7p/18$, $8p/18$ and $9p/18$.

We have

$$c = \frac{4d(n-1)}{\frac{5}{4}n^2 - (n-1)}$$

Let $n = \frac{18}{k}$, where k is 1, 2, 3, 4, 5, 6, 8 & 9.

$$c = \frac{4d\left\{\frac{18}{k} - 1\right\}}{\frac{5}{4}\left(\frac{18}{k}\right)^2 - \left(\frac{18}{k} - 1\right)}$$

$$= \frac{4d\,\frac{(18-k)}{k}}{\frac{5}{4}\frac{18^2}{k^2} - \left(\frac{18}{k} - 1\right)}$$

$$= \frac{4d(18-k)}{\frac{5}{4}\left(\frac{18^2}{k}\right) - k\left(\frac{18}{k} - 1\right)}$$

$$= \frac{4d(18-k)}{\frac{5}{4}\left(\frac{18^2}{k}\right) - \frac{k}{k}(18-k)}$$

$$= \frac{4d.\,k(18-k)}{18 \times \frac{90}{4} - k(18-k)}$$

$$= \frac{4d.\,k(18-k)}{405 - k(18-k)}$$

$$c = \frac{4d.\,k'}{405 - k}\ where\ k' = 18 - k$$

Table 1

Arc length	k	k' = k(18 – K)	405 – k'	C = 4d k' / 405 – k'	Nārāyaṇa's results
50	1	17	388	52 & 56/97	52 & 59/97
100	2	32	373	102 & 354/373	102 & 354/373
150	3	45	360	150	150
200	4	56	349	192 & 192/349	192 & 192/349
250	5	65	340	229 & 7/37	229 & 7/37
300	6	72	333	259 & 17/37	259 & 17/37
350	7	77	328	281 & 29/41	281 & 29/41
400	8	80	325	295 & 5/73	295 & 5/73
450	9	81	324	300	300

The values of *c* are in agreement with Nārāyana's results except for a very small deviation where K=1.

The circumference has been chosen as 900 and the arcs are successively in multiples of 50 making the arcs as *p*/18, *2p/*18, etc to facilitate calculations besides verifying that chord = d when arc length = 1/2 circumference and equals d/2 when arch length = 1/6 circumference.

1.8 An Example from '*Lilavati*'

अष्टादशांशेन समानवृत्तम् एकादिनिघ्नेन च यत्र चापम् ।
पृथक् पृथक् तत्र वदाशु जीवां,खार्कैमितं व्यासदलं च यत्र ॥

(Lī. C XX)

Where the arcs are 1/18, 2/18 etc of the circumference of the circle whose radius is 120, find the length of the chords.

Solution:

d = 240 units.

Arc lengths are 1/18 p, 2/18 p, etc

Chord lengths are calculated and written in the tabular form below.

Table 2

Arc length in terms of Circumference	k	k'= k(18 – k)	405 – k'	C = 4d k' / 405 – k'
P/18	1	17	388	42.06
2P/18	2	32	373	82.35
3P/18	3	45	360	120
4P/18	4	56	349	154.04
5P/18	5	65	340	183.52
6P/18	6	72	333	207.57

7P/18	7	77	328	225.37
8P/18	8	80	325	236.31
9P/18	9	81	324	240

It may be noted that here also the arc to circumference ratios are 1/18, 2/18 etc.[2]

We can evaluate k' and $405 - k'$ separately for k = 1, 2, 3 … up to 9.

2.0 CONCLUSION

The formula for the length of a chord enunciated by Bhāskara has been obtained by means of a dimensional analysis-based approach. A simplified formula for chord length has been obtained. A formula for the chords where the arcs are $p/18$, $2p/18$, etc. has also been derived and applied to two similar examples from *Gaṇita Kaumudī* and *Lilāvatī*. A formula for the side of a regular polygon within a circle has also been derived.

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[2] It is apparent from Table 2 where the circumference is not specified but only the diameter is given, the chord lengths, except for the arc length of p/18 are less than the arc lengths. However, in Table 1 representing the solutions to Narayana's example, the first three values in the table for chord length are larger than the arc lengths. This apparent contradiction may be due to the fact that Π has been assumed as 3 and not 3.1416.